

Urban Distribution Grid Topology Estimation via Group Lasso

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Abstract—The growing penetration of distributed energy resources (DERs) in urban areas raises multiple reliability issues. The topology reconstruction is a critical step to ensure the robustness of distribution grid operation. However, the bus connectivity and network topology reconstruction are hard in distribution grids. The reasons are that 1) the branches are challenging and expensive to monitor due to underground setup; 2) the inappropriate assumption of radial topology in many studies that urban grids are mesh. To address these drawbacks, we propose a new data-driven approach to reconstruct distribution grid topology by utilizing the newly available smart meter data. Specifically, a graphical model is built to model the probabilistic relationships among different voltage measurements. With proof, the bus connectivity and topology estimation problems are formulated as a linear regression problem with least absolute shrinkage on grouped variables (Group Lasso) to deal with meshed network structures. Simulation results show highly accurate estimation in IEEE standard distribution test systems with and without loops using real smart meter data.

I. INTRODUCTION

A key mission of Smart Cities is providing sustainable and economical energy. In these missions, distributed energy resources (DERs) play the key role in sustainable energy and electricity cost reduction. Many countries and cities have proposed their plans to integrate DERs, such as photovoltaics, electric vehicles, and energy storage devices [1]–[3].

Although offering new opportunities, the large-scale integration of DERs also poses new challenges to the distribution system operation. On the generation side, the reverse power flow can render the existing protective systems inadequate. For example, even a small-scale DER integration can destabilize the local grid and cause reliability issues for customers [4]. On the demand side, the frequent plug-and-play electric vehicles will impact the distribution grid power quality such as voltage unbalance and transformer overload [5]. To better control the distribution grid, grid monitoring tools (for islanding and line work hazards) are needed, where topology information (for one or more new buses) is a prerequisite.

The topologies of transmission grids have infrequent changes and can be identified by the topology processor [6], where topology errors are determined by a post-state estimation procedure [7]. Unfortunately, in secondary distribution grids, a topology can frequently change and make existing methods with limited performance. Furthermore, the secondary distribution grids are immense and mostly underground in

many major metropolitan areas. According to [8], in New York City, the total length of underground power cables is about 94000 miles. This fact makes installing any topology identification device time consuming and expensive. Even worse, the reconfiguration of underground distribution grids [9] makes the methods developed for overhead transmission grids unsuitable.

The challenges discussed above make the assumptions in past studies on topology estimation invalid. For instance, [10], [11] assume the availability of the switch connectivity map and search for the right combination. State estimation-based algorithm [12] and power flow-based algorithm [13] require the admittance matrix. These assumptions become inaccurate in urban networks due to the unavailability of switch connectivity maps and admittance matrices in a newly reconfigured distribution network. Even if this information is available, it may be outdated or incorrectly documented due to human interaction without information updating. Also, as many DERs do not belong to the utilities, their connection statuses may not be reported to the utilities. Furthermore, these approaches only focus on radial networks, but many urban secondary distribution grids are mesh [8], [14].

Thankfully, the smart meters at households enable a new opportunity to utilize the time-series data, which are previously unavailable in electric power industry [15], to solve new problems. One application of using these data is reconstructing the distribution grid topology in a data-driven manner because branch data in secondary grids are challenging and expensive to obtain [4] for their underground wiring. Hence, in this paper, we only use end-user data [16], which are easy to obtain. These data are measured by smart meters and include voltage magnitude and real/reactive powers. We aim at helping these devices identify the physical system in which they operate and discover their neighbor buses or local topologies.

In this paper, we firstly prove that a distribution grid can be modeled as a probabilistic graphical model. Then, we propose an algorithm that estimates the connectivity of a bus. This algorithm is based on a linear regression with L_1 penalty on grouped variables, which is known as the group lasso problem. We use the historical data to fit the linear regression. Based on the results of bus connectivity estimation, another type of group lasso method is used to reconstruct network topology when multiple buses have uncertain connectivities. Our algorithm has several advantages. Firstly, it reconstructs a loopy network in a short time. Secondly, unlike other methods [17], our approach does not allow error propagation because the connectivity is recovered at each bus individually. Finally, the proposed algorithm can be implemented with low computational complexity.

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The performance of our data-driven method is verified by simulations on the standard IEEE 8- and 123-bus distribution test cases [18], with and without loops. Real smart meter data collected by Pacific Gas and Electric Company (PG&E) and emulated rooftop solar generation data [19] from National Renewable Energy Laboratory (NREL) are utilized for simulation. Our algorithm outperforms the approaches in recent papers [17], [20] in meshed networks.

The paper is organized as follows: Section II introduces the modeling and the problem of data-driven topology identification. Section III uses a proof to justify that the bus connectivity can be efficiently reconstructed by regularized linear regression. Section IV formulates the network topology reconstruction problem as an optimization problem. Section V evaluates the performance of the new method and Section VI concludes the paper.

II. SYSTEM MODEL

In order to formulate the topology reconstruction problem, firstly, we need to describe the distribution grid and data. A distribution grid is defined as a physical network with branches that connect different buses. To utilize the time series data collected by smart meters, for a M -bus system, we construct a graphical model $G = (\mathcal{S}, \mathcal{E})$ where $\mathcal{S} = \{1, 2, \dots, M\}$ is the set of the vertices and $\mathcal{E} = \{e_{ik}, i, k \in \mathcal{S}\}$ represents the set of the unidirectional edges, i.e., $e_{ik} = e_{ki}$. In our graphical model, a node corresponds to a bus in the physical layer and is modeled as a random variable V_i . The edge that connects node i and k represents the statistical dependence between the measurements collected at bus i and k . The physical network and the graphical model can be visualized in Fig. 1. At time t , the noiseless voltage measurement at bus i is $v_i[t] = |v_i[t]|e^{j\delta_i[t]} \in \mathbb{C}$, where $|v_i[t]| \in \mathbb{R}$ denotes the voltage magnitude in per unit and $\delta_i[t] \in \mathbb{R}$ denotes the voltage phase angle in degree. The measurements are in the steady state and all voltages are sinusoidal signals at the same frequency.

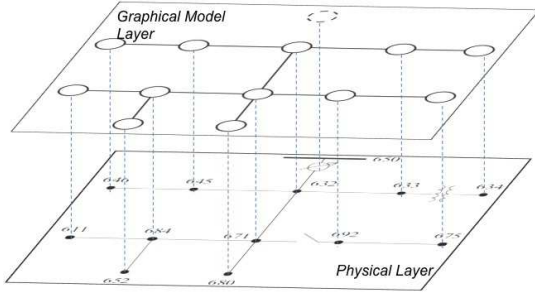


Fig. 1. A physical network with a graphical model layer.

The problem of distribution grid bus connectivity and topology estimation is defined as follows:

- Problem: data-driven topology reconstruction based on voltages
- Given: a sequence of historical measurements $v_i[t], i \in \mathcal{S}, t = 1, \dots, N$ and a partially known grid topology, as shown in Fig. 2
- Find: (1) the connectivity of a bus; (2) the grid topology \mathcal{E} in the dashed box in Fig. 2

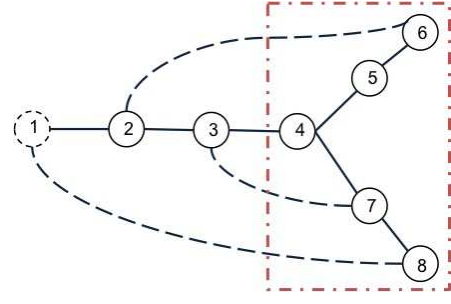


Fig. 2. A graphical model of IEEE 8-bus system with loops. The dashed edges are additional branches to form loops. The branches within the red dashed box are assumed to be unknown.

III. BUS CONNECTIVITY RECONSTRUCTION WITH GROUP LASSO

A. Problem Formulation

In our graphical model, buses are modeled as random variables. Therefore, we use a joint probability distribution to represent the interdependency among buses:

$$\begin{aligned} P(\mathbf{V}_S) &= P(V_2, V_3, \dots, V_M) \\ &= P(V_2)P(V_3|V_2) \cdots P(V_M|V_2, \dots, V_{M-1}). \end{aligned}$$

Bus 1 is the slack bus, which is a constant with unity magnitude and zero phase angle. Therefore, it is omitted from the joint probability distribution above. In a distribution grid, the correlation between interconnected neighboring buses are higher than that between non-neighbor buses. Therefore, a reasonable approximation is

$$P(\mathbf{V}_S) \simeq \prod_{i=2}^M P(V_i | \mathbf{V}_{\mathcal{N}(i)}), \quad (1)$$

where the neighbor set $\mathcal{N}(i)$ contains the vertices of the neighbors of bus i , i.e., $\mathcal{N}(i) = \{k \in \mathcal{S} | e_{ik} \in \mathcal{E}\}$. With this approximation, finding the connectivity of a bus is equivalent to finding the neighbors of the bus. Notice that we are not restricted to parent nodes here because the topology in a secondary distribution grid can be both radial and mesh [20]. We aim to propose an algorithm that works for both types of networks.

Voltage measurements usually have an irregular probability distribution and are hard to be used for our goal in this paper. Therefore, instead of using voltage measurements directly, we use the incremental change of the voltage measurements to reconstruct the distribution grid topology [21]. The increment change of voltage at bus i is defined as $\Delta v_i[t] = v_i[t] - v_i[t-1]$ and $\Delta v_i[1] = 0$. We use ΔV_i to represent the random variable of voltage change at bus i . As shown in Fig. 3, ΔV_i follows a Gaussian distribution approximately and empirically.

With ΔV_i , the probability distribution in (1) becomes

$$P(\Delta \mathbf{V}_S) \simeq \prod_{i=2}^M P(\Delta V_i | \Delta \mathbf{V}_{\mathcal{N}(i)}). \quad (2)$$

In the following, we will show that, with an appropriate assumption, ΔV_i only has statistical dependence with its neighbors. We will also explain why we can approximate $P(\Delta \mathbf{V}_S)$ in such a way. In the following context, the operator \setminus denotes the complement operator, i.e., $\mathcal{A} \setminus \mathcal{B} = \{i \in \mathcal{A}, i \notin \mathcal{B}\}$.

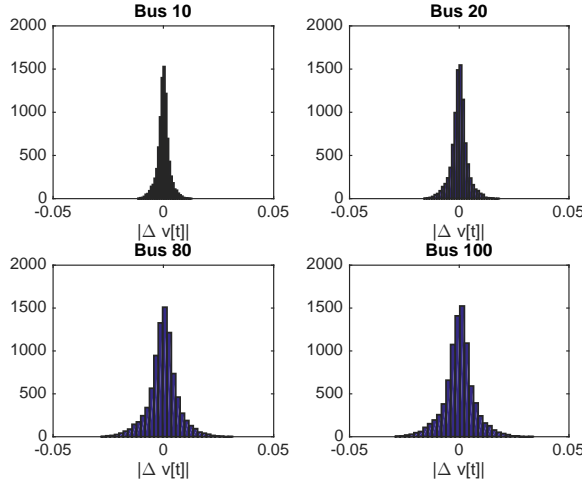


Fig. 3. Histogram of $|\Delta v[t]|$ of four buses in IEEE 123-bus system.

Theorem 1. *In a distribution grid, if the change of the current injection ΔI at each bus is approximately independent, the voltage change of bus i and the voltage changes of all other buses that are not connected with bus i are conditionally independent, given the voltage changes of the neighbors of bus i , i.e., $\Delta V_i \perp \{\Delta V_d, d \in \mathcal{S} \setminus \{\mathcal{N}(i), i\}\} | \Delta \mathbf{V}_{\mathcal{N}(i)}$.*

Proof: In this proof, we will firstly show the conditional independence for a simple example. Then, we will prove the conditional independence in generalized distribution networks.

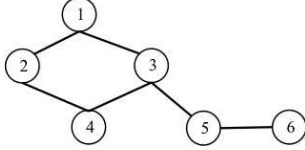


Fig. 4. A 6-bus system.

For the network in Fig. 4, the circuit equation $\mathbf{Y}\Delta\mathbf{V} = \Delta\mathbf{I}$ is equivalent to:

$$\begin{bmatrix} y_{11} & -y_{12} & -y_{13} & 0 & 0 & 0 \\ -y_{12} & y_{22} & 0 & -y_{24} & 0 & 0 \\ -y_{13} & 0 & y_{33} & -y_{34} & -y_{35} & 0 \\ 0 & -y_{24} & -y_{34} & y_{44} & 0 & 0 \\ 0 & 0 & -y_{35} & 0 & y_{55} & -y_{56} \\ 0 & 0 & 0 & 0 & -y_{56} & y_{66} \end{bmatrix} \begin{bmatrix} \Delta V_1 \\ \Delta V_2 \\ \Delta V_3 \\ \Delta V_4 \\ \Delta V_5 \\ \Delta V_6 \end{bmatrix} = \begin{bmatrix} \Delta I_1 \\ \Delta I_2 \\ \Delta I_3 \\ \Delta I_4 \\ \Delta I_5 \\ \Delta I_6 \end{bmatrix}$$

where $y_{ik} = y_{ki}$ denotes the deterministic admittance between bus i and bus k , and the self-admittance is defined as $y_{ii} = \sum_{k=1, i \neq k}^6 y_{ik}$. If $y_{ik} = 0$, there is no branch between bus i and k .

For bus 1, the neighbor set $\mathcal{N}(1)$ is $\{2, 3\}$. Given $\Delta V_2 = \Delta v_2$ and $\Delta V_3 = \Delta v_3$, we have following equations:

$$\Delta I_1 + \Delta v_2 y_{12} + \Delta v_3 y_{13} = \Delta V_1 y_{11} \quad (3)$$

$$\Delta I_4 + \Delta v_2 y_{24} + \Delta v_3 y_{34} = \Delta V_4 y_{44} \quad (4)$$

$$\Delta I_5 + \Delta v_3 y_{35} = \Delta V_5 y_{55} - \Delta V_6 y_{56} \quad (5)$$

$$\Delta I_6 = -\Delta V_5 y_{56} + \Delta V_6 y_{66} \quad (6)$$

For bus 4, due to the assumption of current injection independence, i.e., $\Delta I_1 \perp \Delta I_4$, ΔV_1 and ΔV_4 are conditionally independent given ΔV_2 and ΔV_3 . Similarly, for bus 5 and bus 6, $\Delta V_1 | \{\Delta V_2, \Delta V_3\}$ and $\{\Delta V_5, \Delta V_6\} | \{\Delta V_2, \Delta V_3\}$ are independent because the current incremental change at each bus is independent with others. Therefore, for this example, we prove that the conditional independence holds.

Let's extend the proof to a more general case. For a grid with M buses, the current and voltage relationship of bus i can be written as

$$\Delta I_i = \Delta V_i y_{ii} - \sum_{k \in \mathcal{N}(i)} \Delta V_k y_{ik}, \quad (7)$$

with $y_{ii} = \sum_{k \in \mathcal{N}(i)} y_{ik}$. Given $\Delta V_k = \Delta v_k$ for all $k \in \mathcal{N}(i)$, the equation above becomes $\Delta I_i + \sum_{k \in \mathcal{N}(i)} \Delta v_k y_{ki} = \Delta V_i y_{ii}$.

For bus d , which is not connected with bus i , i.e., $d \in \mathcal{S} \setminus \{\mathcal{N}(i), i\}$, we have a similar equation:

$$\Delta I_d + \sum_{k \in \mathcal{N}(d)} \Delta V_k y_{dk} = \Delta V_d y_{dd}. \quad (8)$$

The relationship between the neighbor sets $\mathcal{N}(i)$ and $\mathcal{N}(d)$ can be divided into two scenarios.

(1) When $\mathcal{N}(i) \cap \mathcal{N}(d) = \emptyset$, (8) remains the same. Therefore, ΔV_i and $\{\Delta V_d, \Delta \mathbf{V}_{\mathcal{N}(d)}\}$ are conditionally independent given $\Delta \mathbf{V}_{\mathcal{N}(i)}$. This case is similar to ΔV_6 in (6).

(2) When $\mathcal{N}(i) \cap \mathcal{N}(d) \neq \emptyset$, (8) becomes

$$\Delta I_d + \sum_{k \in \mathcal{N}(i) \cap \mathcal{N}(d)} \Delta v_k y_{dk} = \Delta V_d y_{dd} - \sum_{k \in \mathcal{N}(d) \setminus \mathcal{N}(i)} \Delta V_k y_{dk}.$$

Hence, since the current incremental change ΔI is independent at each bus, ΔV_i and $\{\Delta V_d, \Delta \mathbf{V}_{\mathcal{N}(d) \setminus \mathcal{N}(i)}\}$ are independent conditioning on $\Delta \mathbf{V}_{\mathcal{N}(i)}$. Therefore, the conditional independence is proved in this case. This case is similar to ΔV_5 in (5). Also, ΔV_4 in (4) is a special case where $\mathcal{N}(d) = \mathcal{N}(i)$. In conclusion, given $\Delta \mathbf{V}_{\mathcal{N}(i)}$, ΔV_i and $\Delta \mathbf{V}_{\mathcal{S} \setminus \{\mathcal{N}(i), i\}}$ are conditionally independent. ■

With Theorem 1, we can claim that the voltage change at each bus only depends on its neighbors, i.e., $P(\Delta \mathbf{V}_{\mathcal{S}}) = \prod_{i=2}^M P(\Delta V_i | \Delta \mathbf{V}_{\mathcal{N}(i)})$. In the following section, we will propose an efficient algorithm to find $\mathcal{N}(i)$.

B. Bus Connectivity Reconstruction via Linear Regression

For bus i , $\Delta \mathbf{V}_{\mathcal{S} \setminus \{i\}}$ denotes the collection of all random variables in the graph beside V_i . If we assume $\Delta \mathbf{V}_{\mathcal{S}}$ to follow a multivariate Gaussian distribution, as empirically shown in Fig. 3, the conditional distribution of ΔV_i given $\Delta \mathbf{V}_{\mathcal{S} \setminus \{i\}}$ also follows a Gaussian distribution.

Therefore, based on the Gaussian probability density function, ΔV_i can be described by a linear equation based on $\Delta \mathbf{V}_{\mathcal{S} \setminus \{i\}}$ and an error term, as shown below:

$$\Delta V_i = \Delta \mathbf{V}_{\mathcal{S} \setminus \{i\}}^T \boldsymbol{\beta}^{(i)} + \epsilon_{\mathcal{S} \setminus \{i\}}, \quad (9)$$

where $\boldsymbol{\beta}^{(i)}$ denotes the parameter vector and T denotes the transpose operator. In this linear equation, $\epsilon_{\mathcal{S} \setminus \{i\}}$ is a zero-mean Gaussian variable with $\text{Var}(\epsilon_{\mathcal{S} \setminus \{i\}}) = \text{Var}(\Delta V_i | \Delta \mathbf{V}_{\mathcal{S} \setminus \{i\}})$ and is independent with $\Delta \mathbf{V}_{\mathcal{S} \setminus \{i\}}$ [22]. In Theorem 1, we prove that the voltage changes of two buses are conditionally independent if these two buses are not connected. Therefore, a nonzero coefficient in $\boldsymbol{\beta}^{(i)}$ indicates

the statistical dependence between two nodes. Hence, the connectivity identification problem becomes a linear regression problem. For each bus i , we can use the estimated parameter vector $\hat{\beta}^{(i)}$ to find its neighbors.

From physical layer perspective, we can also show that the nonzero coefficients in the parameter vector $\beta^{(i)}$ indicate the bus connectivity. Specifically, rewriting (7), we have

$$\begin{aligned}\Delta I_i &= \Delta V_i y_{ii} - \sum_{k \in \mathcal{N}(i)} \Delta V_k y_{ik}, \\ \Delta V_i &= \frac{\Delta I_i}{y_{ii}} + \sum_{k \in \mathcal{N}(i)} \frac{y_{ik}}{y_{ii}} \Delta V_k.\end{aligned}\quad (10)$$

Compared with (9), we can observe that if bus k connects with bus i , i.e., $k \in \mathcal{N}(i)$, the k -th element of $\beta^{(i)}$ is y_{ik}/y_{ii} . If bus d is not an element of the set $\mathcal{N}(i)$, the d -th element of $\beta^{(i)}$ is zero. The reason is that $y_{id} = 0$ and the voltage changes are conditionally independent, as proved by Theorem 1. The variation introduced by ΔI_i is captured by $\epsilon_{\mathcal{S} \setminus \{i\}}$. If we assume ΔV has a zero mean, $\epsilon_{\mathcal{S} \setminus \{i\}}$ also has a zero mean and follows a Gaussian distribution since $\Delta V_{\mathcal{S}}$ follows a multivariate Gaussian distribution, which is consistent with our previous discussion.

A typical distribution grid is not fully connected. Therefore, the graph is sparse and many coefficients in $\beta^{(i)}$ are zero. A widely used constraint to ensure the sparsity is L_1 norm because it leads to a convex optimization problem and can be solved efficiently. This type of problem is also known as *Lasso* [23]. It minimizes the sum of squared errors with a bound on the sum of the absolute values of parameters (L_1 norm). With L_1 norm penalty and N measurements, the linear regression in (9) is formulated as:

$$\hat{\beta}^{(i)} = \arg \min_{\beta^{(i)}} \sum_{t=1}^N \left(\Delta v_i[t] - \sum_{\substack{k=2 \\ k \neq i}}^M \Delta v_k[t] \beta_k^{(i)} \right)^2 + \lambda \|\beta^{(i)}\|_1, \quad (11)$$

where $\lambda \geq 0$ is the regularization parameter and $\|\cdot\|_1$ is L_1 norm. The term $\|\beta^{(i)}\|_1$ is called the penalty term. If $\lambda = 0$, the optimization problem in (11) is a standard least squares problem. After solving the lasso problem, we can find the neighbor set $\hat{\mathcal{N}}^{(i)}$ which contains the indices of all non-zero coefficients in $\hat{\beta}^{(i)}$.

C. Bus Neighbors Estimation via Group Lasso

In the previous section, we have formulated the bus connectivity estimation problem as a lasso problem. However, this formulation is hard to be solved by utilizing many well-known lasso solvers. The reason is that many existing approaches only solve lasso problem with real numbers. However, in power systems, ΔV and the parameter β are complex numbers. To address this issue, we propose a group lasso approach by firstly converting a complex number lasso formulation to a real number lasso problem.

For two arbitrary complex numbers x and y , their product $z = xy$ is expressed as

$$\text{Re}(z) = \text{Re}(x) \text{Re}(y) - \text{Im}(x) \text{Im}(y), \quad (12)$$

$$\text{Im}(z) = \text{Re}(x) \text{Im}(y) + \text{Im}(x) \text{Re}(y). \quad (13)$$

Thus, the linear equation in (9) can be rewritten as

$$\begin{aligned}\mathbf{D}_i &= \mathbf{Z} \gamma^{(i)} \\ \begin{bmatrix} \text{Re}(\Delta V_i) \\ \text{Im}(\Delta V_i) \end{bmatrix} &= \begin{bmatrix} \text{Re}(\Delta V_2) & \text{Im}(\Delta V_2) \\ -\text{Im}(\Delta V_2) & \text{Re}(\Delta V_2) \\ \vdots & \vdots \\ \text{Re}(\Delta V_{i-1}) & \text{Im}(\Delta V_{i-1}) \\ -\text{Im}(\Delta V_{i-1}) & \text{Re}(\Delta V_{i-1}) \\ \text{Re}(\Delta V_{i+1}) & \text{Im}(\Delta V_{i+1}) \\ -\text{Im}(\Delta V_{i+1}) & \text{Re}(\Delta V_{i+1}) \\ \vdots & \vdots \\ \text{Re}(\Delta V_M) & \text{Im}(\Delta V_M) \\ -\text{Im}(\Delta V_M) & \text{Re}(\Delta V_M) \end{bmatrix}^T \begin{bmatrix} \text{Re}(\beta_2^{(i)}) \\ \text{Im}(\beta_2^{(i)}) \\ \vdots \\ \text{Re}(\beta_{i-1}^{(i)}) \\ \text{Im}(\beta_{i-1}^{(i)}) \\ \text{Re}(\beta_{i+1}^{(i)}) \\ \text{Im}(\beta_{i+1}^{(i)}) \\ \vdots \\ \text{Re}(\beta_M^{(i)}) \\ \text{Im}(\beta_M^{(i)}) \end{bmatrix} \\ &= \sum_{\substack{k=2 \\ k \neq i}}^M \begin{bmatrix} \text{Re}(\Delta V_k) & -\text{Im}(\Delta V_k) \\ \text{Im}(\Delta V_k) & \text{Re}(\Delta V_k) \end{bmatrix} \begin{bmatrix} \text{Re}(\beta_k^{(i)}) \\ \text{Im}(\beta_k^{(i)}) \end{bmatrix}, \\ \mathbf{D}_i &= \sum_{\substack{k=2 \\ k \neq i}}^M \mathbf{z}_k \gamma_k^{(i)}.\end{aligned}\quad (14)$$

In (14), we transform a complex linear equation to a real linear equation. We can apply the L_1 constraint to the new parameter vector $\gamma^{(i)}$, which becomes a real lasso problem.

Solving the linear regression in (14) with L_1 penalty $\|\gamma^{(i)}\|_1$ results a sparse estimate $\hat{\gamma}^{(i)}$. However, we cannot guarantee that both $\text{Re}(\beta_k^{(i)})$ and $\text{Im}(\beta_k^{(i)})$ are zero or nonzero at the same time. If bus k is not connected with bus i , $\beta_k^{(i)}$ is zero in (11). Thus, both $\text{Re}(\beta_k^{(i)})$ and $\text{Im}(\beta_k^{(i)})$ are zeros. To enforce the sparsity on both real and imaginary parts of β , in (14), we need to apply sparsity constraint to a group of elements in $\gamma^{(i)}$ such that all elements within a group will be zero if one of them is zero. This type of problem is known as *Group Lasso* [24]. Particularly, we can estimate $\gamma^{(i)}$ as follows:

$$\hat{\gamma}^{(i)} = \arg \min_{\gamma^{(i)}} \sum_{t=1}^N \left(\mathbf{d}_i[t] - \sum_{\substack{k=2 \\ k \neq i}}^M \mathbf{z}_k[t] \gamma_k^{(i)} \right)^2 + \lambda \sum_{\substack{k=2 \\ k \neq i}}^M \|\gamma_k^{(i)}\|_2. \quad (15)$$

Unlike the lasso formulation in (11), in the group lasso (15), we use L_2 norm because it enforces the entire vector $\gamma_k^{(i)}$ to be zero or nonzero. For more discussion on the group lasso, see [22] and [25].

We can construct $\hat{\beta}^{(i)}$ from $\hat{\gamma}^{(i)}$ and find the indices of non-zero elements in $\hat{\beta}^{(i)}$. Alternatively, if bus k is a neighbor of bus i , both elements in $\hat{\gamma}^{(i)}$ are zero.

In distribution grids, phase angles are hard to obtain because the phasor measurement units (PMUs) have not been widely installed. When only the change of voltage magnitude $|\Delta V_i|$ is available, \mathbf{D}_i and \mathbf{Z} become $|\Delta V_i|$ and $[|\Delta V_2|, \dots, |\Delta V_M|]$ respectively. Also, $\gamma_k^{(i)}$ reduces to a scalar. The group lasso

problem in (15) can be rewritten as

$$\begin{aligned} & \sum_{t=1}^N \left(\left| \Delta v_i[t] - \sum_{\substack{k=2 \\ k \neq i}}^M |\Delta v_k[t]| \gamma_k^{(i)} \right| \right)^2 + \lambda \sum_{\substack{k=2 \\ k \neq i}}^M \|\gamma_k^{(i)}\|_2 \\ &= \sum_{t=1}^N \left(\left| \Delta v_i[t] - \sum_{\substack{k=2 \\ k \neq i}}^M |\Delta v_k[t]| \gamma_k^{(i)} \right| \right)^2 + \lambda \|\gamma^{(i)}\|_1. \end{aligned} \quad (16)$$

In this new formulation, $\|\gamma_k^{(i)}\|_2$ is equivalent to $|\gamma_k^{(i)}|$ and $\sum_{k=2, k \neq i}^M \|\gamma_k^{(i)}\|_2 = \sum_{k=2, k \neq i}^M |\gamma_k^{(i)}| = \|\gamma^{(i)}\|_1$, where $\gamma^{(i)} \in \mathbb{R}^{M-2}$. (16) becomes the ordinary lasso problem formulation. With voltage magnitude measurements only, we can reconstruct the bus connectivity using the ordinary lasso problem formulation. Also, $\hat{\beta}^{(i)}$ is the same as $\hat{\gamma}^{(i)}$.

The steps of our proposed algorithm for bus connectivity reconstruction are summarized in Algorithm. 1.

Algorithm 1 Distribution Grid Bus Connectivity Reconstruction via Group Lasso

Require: $\Delta v_i[t]$ for $t = 1, \dots, N$

- 1: Normalize and standardize ΔV_i such that it has a zero mean and an unity variance on real and imaginary parts.
 - 2: Solve the group lasso optimization problem in (15) for bus i and find the parameter vector estimate $\hat{\gamma}^{(i)}$.
 - 3: Estimate $\hat{\beta}^{(i)}$ as $\hat{\beta}_k^{(i)} = \hat{\gamma}_{k,1}^{(i)} + j\hat{\gamma}_{k,2}^{(i)}$.
-

D. Choice of Regularization Parameter

The choice of λ is critical in lasso type problems because it affects the number of non-zero coefficients in $\beta^{(i)}$. When λ is small, the penalty term has no effect and the solution is close to the original least squares solution. When λ is large, some coefficients of $\hat{\beta}^{(i)}$ are zeros. A well-known criterion to choose the parameter λ is Bayesian information criterion (BIC) from statistics. For bus i , the BIC is defined as

$$\text{BIC}_i(\lambda) = \tilde{N} \ln \left(\frac{\text{RSS}_i(\lambda)}{\tilde{N}} \right) + \ln(\tilde{N}) \times \hat{k}, \quad (17)$$

where \hat{k} denotes the number of non-zero elements in $\hat{\beta}^{(i)}$ [26]. The residual sum of squares (RSS) is defined as

$$\text{RSS}_i(\lambda) = \sum_{t=1}^N \left(y_i[t] - \sum_{\substack{k=2 \\ k \neq i}}^M z_k[t] \hat{\gamma}_k^{(i)} \right)^2.$$

\tilde{N} is $2N$ for the group lasso in (15) or N for the lasso problem in (16). We select λ that minimizes $\text{BIC}_i(\lambda)$ as the optimal tuning parameter for bus i .

IV. GRID TOPOLOGY RECONSTRUCTION VIA GROUP LASSO

In the previous section, we have shown to use group lasso method to reconstruct the connectivity of a bus in distribution grids. In many scenarios, a subnetwork, which contains multiple buses, has unknown topology. In this section, we will extend the presented method from a bus to a network.

When we solve the group lasso problem for each bus i , the result is a neighbor set $\mathcal{N}(i)$. If bus i and bus k are connected, bus i is a neighbor of bus k and vice versa. In (15), we find the neighbors of a single bus. Therefore, an edge between bus i and k is visited twice because each bus has an individual estimate $\hat{\beta}$. A simple way to combine both estimates is AND rule, $\hat{e}_{ik}^{\text{AND}} = \hat{\beta}_k^{(i)} \wedge \hat{\beta}_i^{(k)}$, where \wedge denotes the logical “and” operator. We can apply AND rule to every pair of buses within the unknown subnetwork and estimate the topology at last.

The AND rule provides reliable results but also has a high probability to miss an edge. This is because an edge is decided only when both estimates have an agreement. If one estimate has an error, an edge will be missed. To address this drawback, we propose another rule called OR rule, $\hat{e}_{ik}^{\text{OR}} = \hat{\beta}_k^{(i)} \vee \hat{\beta}_i^{(k)}$, where \vee denotes the logical “or” operator.

A disadvantage of both AND rule and OR rule is that they cannot guarantee that the estimated graph is a connected graph. It means that some buses form a small graph and are isolated from other buses. To overcome this drawback, we use both rules to decide an edge. We propose a new rule, AND-OR rule, to merge the results of AND and OR rules. This new rule has multiple steps. Firstly, the AND rule is applied and produces an estimated edge set $\hat{\mathcal{E}}_{\text{AND}}$. Then, we apply a physical law to diagnose the estimated edges. In power systems, differences in voltage magnitudes drive the flow of reactive power [27]. Therefore, unless it is a substation, bus i should have at least one neighbor with higher voltage magnitude because the reactive power needs to be supplied from a bus with higher voltage magnitude. If none of the neighbors of bus i has higher $|V|$, we can conclude that bus i is not connected to the main grid. Therefore, we apply the OR rule with some modifications to find a new neighbor bus k , as shown below:

$$\hat{e}_{ik} = (\hat{\beta}_k^{(i)} \vee \hat{\beta}_i^{(k)}) \wedge \neg(\hat{\beta}_k^{(i)} \wedge \hat{\beta}_i^{(k)}) \wedge (\hat{E}(|V_k|) > \hat{E}(|V_i|)), \quad (18)$$

where \neg denotes the logical “not” operator and \hat{E} denotes the empirical mean. The estimated edge set is denoted as $\hat{\mathcal{E}}_{\text{AND-OR}}$. These steps are summarized in a flow chart in Fig. 5. In Section V, we will show that the AND-OR rule is more robust than the AND or OR rule.

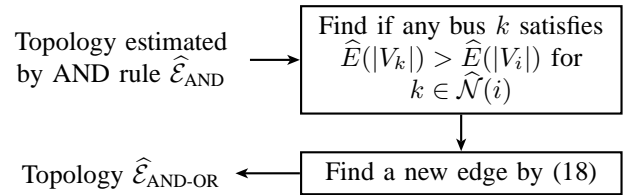


Fig. 5. Flow chart of AND-OR rule.

A. Improvement

In the section above, we proposed to use the group lasso and AND-OR rule to reconstruct the distribution grid topology. However, this approach requires an extra step, the logical rule combination, to recover the topology. Also, this approach requires to solve multiple optimization problems and cannot utilize the interactions between bus measurements. In this

section, we propose another approach that reconstructs the topology as a single optimization problem and better utilizes data.

Let's start with a simple case where we only have voltage magnitude data. In a 3-bus system, we have following linear equations:

$$\begin{aligned} |\Delta V_1| &= |\Delta V_2| \tilde{\beta}_2^{(1)} + |\Delta V_3| \tilde{\beta}_3^{(1)} \\ |\Delta V_2| &= |\Delta V_1| \tilde{\beta}_1^{(2)} + |\Delta V_3| \tilde{\beta}_3^{(2)} \\ |\Delta V_3| &= |\Delta V_1| \tilde{\beta}_1^{(3)} + |\Delta V_2| \tilde{\beta}_2^{(3)}. \end{aligned}$$

These equations can be written in a matrix format, e.g.,

$$\begin{bmatrix} |\Delta V_1| \\ |\Delta V_2| \\ |\Delta V_3| \end{bmatrix} = \begin{bmatrix} |\Delta V_2| & 0 & 0 \\ 0 & |\Delta V_1| & 0 \\ |\Delta V_3| & 0 & 0 \\ 0 & 0 & |\Delta V_1| \\ 0 & |\Delta V_3| & 0 \\ 0 & 0 & |\Delta V_2| \end{bmatrix}^T \begin{bmatrix} \tilde{\beta}_2^{(1)} \\ \tilde{\beta}_1^{(2)} \\ \tilde{\beta}_3^{(1)} \\ \tilde{\beta}_1^{(3)} \\ \tilde{\beta}_3^{(2)} \\ \tilde{\beta}_2^{(3)} \end{bmatrix}$$

$$|\Delta \mathbf{V}| = \mathbf{P} \tilde{\boldsymbol{\beta}}. \quad (19)$$

Reconstructing the distribution grid topology by AND or OR rule is equivalent to solving the linear system in (19) with different penalties. Specifically, for a M -bus system, the following optimization problem is equivalent to OR rule:

$$\hat{\boldsymbol{\beta}}^{\text{OR}} = \arg \min_{\tilde{\boldsymbol{\beta}}} \sum_{t=1}^N \left(|\Delta \mathbf{v}[t]| - \mathbf{P}[t] \tilde{\boldsymbol{\beta}} \right)^2 + \lambda \|\tilde{\boldsymbol{\beta}}\|_1, \quad (20)$$

where $|\Delta \mathbf{v}[t]| \in \mathbb{R}^{M-1 \times 1}$, $\mathbf{P} \in \mathbb{R}^{(M-1) \times (M-1)(M-2)}$, and $\tilde{\boldsymbol{\beta}} \in \mathbb{R}^{(M-1)(M-2) \times 1}$. With L_1 norm penalty, any element in $\tilde{\boldsymbol{\beta}}$ can be zero. Therefore, (20) is equivalent to OR rule.

For AND rule, $\tilde{\beta}_k^{(i)}$ and $\tilde{\beta}_i^{(k)}$ are either zero or non-zero at the same time. Therefore, we can use the group lasso to solve problem in (19). In details, we can solve the following optimization problem:

$$\hat{\boldsymbol{\beta}}^{\text{AND}} = \arg \min_{\tilde{\boldsymbol{\beta}}} \sum_{t=1}^N \left(|\Delta \mathbf{v}[t]| - \mathbf{P}[t] \tilde{\boldsymbol{\beta}} \right)^2 + \lambda \sum_{\substack{i,k=2 \\ k \neq i}}^M \|(\tilde{\beta}_k^{(i)}, \tilde{\beta}_i^{(k)})\|_2. \quad (21)$$

In (21), we put every two buses into a group and use penalty term to decide if these two buses are connected. Therefore, (21) is equivalent to AND rule. We can use the similar method in Section III-D to choose the optimal λ .

V. SIMULATION AND RESULTS

The simulations are implemented on the IEEE PES distribution networks for IEEE 8-bus and 123-bus networks [18]. In order to validate the performance of the proposed approach on loopy networks, we add several branches in both networks to create loops. The modified loopy 8-bus system is shown in Fig. 2. In each network, bus 1 is selected as the slack bus. The historical data have been preprocessed by the MATLAB Power System Simulation Package (MATPOWER) [28]. To simulate the power system behavior in a practical pattern, we use the load profile from PG&E in the subsequent simulation. This profile contains anonymized and secure hourly smart meter readings over 110,000 PG&E residential customers for a period of one year spanning from 2011 to 2012. The reactive

power $q_i[t]$ at bus i and time t is computed according to a randomly generated power factor $pf_i[t]$, which follows a uniform distribution, e.g., $pf_i[t] \sim \text{Unif}(0.85, 0.95)$. To obtain time-series voltage magnitude at time, i.e., $|v_i[t]|$, we run a power flow to generate the hourly states of the power system over a year. $N = 8760$ measurements are obtained at each bus.

A. Error of Bus Connectivity Reconstruction

For bus i , we define the connectivity error as

$$\text{Error}(i) = \sum_{k \in \mathcal{N}(i)} \mathbb{I}(k \notin \hat{\mathcal{N}}(i)) + \sum_{k \in \hat{\mathcal{N}}(i)} \mathbb{I}(k \notin \mathcal{N}(i)),$$

where $\mathbb{I}(\cdot)$ denotes the indicator function. The first part represents the number of missing neighbors and the second part represents the number of incorrect neighbors.

For 8-bus networks, with or without loops, our algorithm presented in Section III achieves zero error. Fig. 6 shows the error at each bus for 123-bus networks, with or without loops. We can observe that most buses have zero error. Only two or three buses incorrectly find one neighbor.

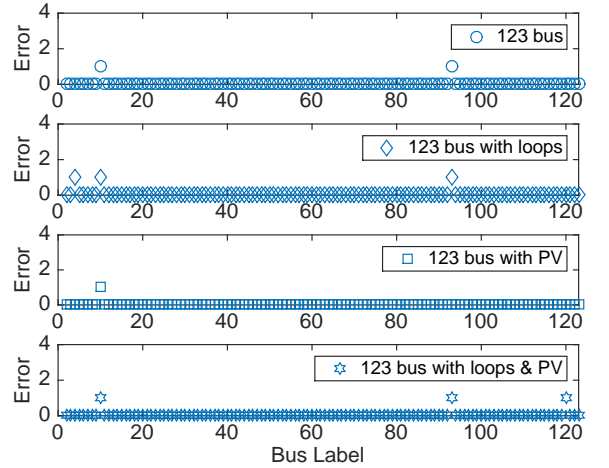


Fig. 6. Errors of bus connectivity reconstruction for different networks using $|\mathbf{V}|$ only; $\lambda = 0.1$.

B. Network Topology Reconstruction Error Rate

To summarize performances in various simulation cases, we define the error rate (ER) as

$$\text{ER} = \left(\frac{\sum_{e_{ij} \in \mathcal{E}} \mathbb{I}(e_{ij} \notin \hat{\mathcal{E}}) + \sum_{e_{ij} \in \hat{\mathcal{E}}} \mathbb{I}(e_{ij} \notin \mathcal{E})}{|\mathcal{E}|} \right) \times 100\%$$

where $\hat{\mathcal{E}}$ denotes the estimated set of branches and $|\mathcal{E}|$ denotes the size of the set \mathcal{E} . The first term of ER counts the number of missing edges. The second term counts the number of falsely connected edges.

Table I summarizes ER on 8-bus and 123-bus systems with different available data. When the system has PMU installed, the group lasso method reconstructs the system with zero error. When only voltage magnitude is available, the performance of AND rule is degraded.

TABLE I. NETWORK TOPOLOGY RECONSTRUCTION ERROR RATE WITHOUT DERs, $\lambda = 0.1$

| Rule & Available Data | 8-Bus | 8-Bus with Loops | 123-Bus | 123-Bus with Loops |
|-----------------------|--------|------------------|---------|--------------------|
| AND, $ V , \theta$ | 0% | 0% | 0% | 0% |
| AND, $ V $ | 0% | 20% | 4.07% | 4.07% |
| OR, $ V $ | 14.29% | 10% | 2.44% | 1.63% |
| AND-OR, $ V $ | 0% | 0% | 0% | 0% |

Table I also summarizes ER of different rules. The AND-OR rule can perfectly recover the topologies in all four networks. For the OR rule, it incorrectly detects one edge in 8-bus networks. The AND rule makes no mistake in 8-bus radial network but one error in 8-bus loop network. For 123-bus networks, the OR rule outperforms the AND rule.

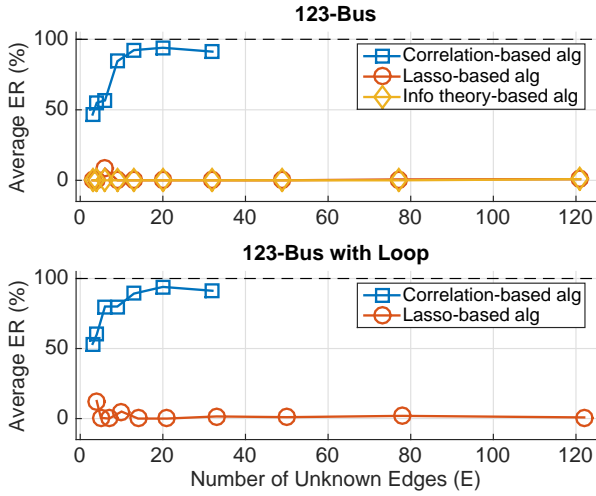


Fig. 7. ER comparison of the proposed algorithm and other existing methods in IEEE 123-bus system using $|V|$, with and without loop.

In Fig. 7, we compare the proposed algorithm with a correlation-based algorithm [17] and an information theory-based algorithm [20]. The optimization problem in [17] is solved by CVX, a package for specifying and solving convex programs [29]. The x -coordinate represents the number of edges that are needed to be identified. The y -coordinate represents the average error rate over 100 iterations. Our algorithm consistently has an approximate zero error rate, which is comparable with the approach in [20] in a radial network, while the detection ability of [17] decreases. The information theory-based algorithm is excluded in the loop network because it is proposed for radial networks.

C. Networks with DER Integration

All buses in 8-bus networks and 12 buses in 123-bus networks are integrated with solar panels. The hourly power generation profile is computed by PVWatts Calculator, an online application developed by NREL that estimates the power generation of a photovoltaic system based on weather and physical parameters [19]. The data are computed based on the weather history in North California and the physical parameters of a 5kW solar panel. The renewable power generator is modeled as a negative load.

For bus connectivity estimation, in 8-bus networks, the proposed approach find the bus' neighbors without any error. In 123-bus networks, as shown in Fig. 6, our approach has nearly perfect performance.

TABLE II. NETWORK TOPOLOGY RECONSTRUCTION ERROR RATE WITH DER INTEGRATION USING $|V|$, $\lambda = 0.1$

| Rule | 8-Bus | 8-Bus with Loops | 123-Bus | 123-Bus with Loops |
|--------|--------|------------------|---------|--------------------|
| AND | 14.29% | 10% | 2.44% | 4.07% |
| OR | 14.29% | 10% | 0.81% | 0% |
| AND-OR | 0% | 0% | 0% | 0% |

Table II illustrates the error rates of grid topology reconstruction with the integration of DERs. The AND-OR rule still has the lowest error rate and perfectly recovers the topologies in all four networks. In 8-bus network with loops, both the AND and OR rules have zero error as well. Unlike the case without DERs, both AND rule and OR rule have better performances in 123-bus networks.

D. Sensitivity to Historical Data Length

To explore the sensitivity of the proposed algorithm to the sample number, we run Monte Carlo simulations by using data from 2 days to 365 days. Fig. 8 compares the error rates of AND and AND-OR rules with different data lengths. We can see that with around 100 days' measurements ($100 \times 24 = 2400$ data points), AND-OR rule can achieve zero error. If the measurement is sampled with higher sampling frequencies, e.g., every 15 minutes, our algorithm only requires 25 days' measurements to perfectly reconstruct the grid topology. This result reflects that our algorithm can provide robust reconstruction with a short period of historical data.

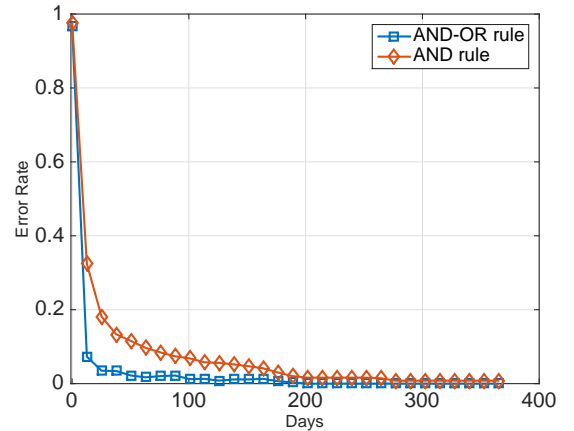


Fig. 8. Comparison of error rates with different size of available data of 123-bus loop network using $|V|$.

E. Computational Complexity

Fig. 9 compares the computational time of our approach with the correlation-based algorithm in loop networks. The time is averaged over 20 iterations. All computations were done on a personal computer with Intel(R) Core(TM) i7

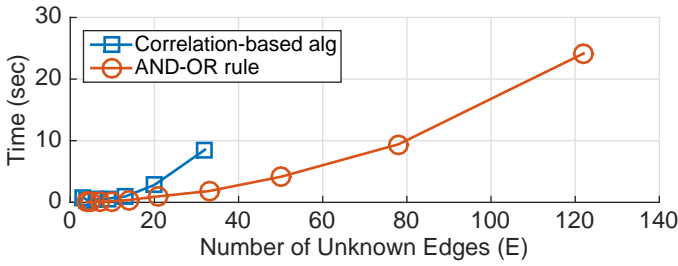


Fig. 9. Comparison of computational time of the proposed algorithm and other existing methods in IEEE 123-bus system with loops.

3615QM CPU @ 2.30GHz. Fig. 9 shows our method is consistently faster than [17]. For 8-bus network with loops in Fig. 2, our approach requires 0.3 seconds. Although [20] mainly focuses on the radial topology, it can be extended to find one loop by running the tree-based algorithm for each possible 3-tuple of buses, which is $8 \times 7 \times 6 = 336$ combinations. With 0.1 seconds per run, the time to reconstruct a 8-bus system with one loop is 33.6 seconds, much slower than our method. Even worse, the system in Fig. 2 has three loops and the required time of [20] will increase exponentially. Hence, our approach is the most efficient one in mesh networks.

VI. CONCLUSION

We present a data-driven algorithm that reconstructs the bus connectivity and grid topology in secondary distribution grids. Unlike existing approaches, our method does not require the knowledge about the switch connectivity maps or admittance matrices and only utilizes the newly available smart meter data. Also, many past studies only focus on the radial topology but our method can reconstruct both radial and mesh networks. We prove that a distribution grid can be formulated as a probabilistic graphical model and be reconstructed by group lasso. We validate the proposed algorithm on both IEEE 8- and 123-bus systems with and without loops using real data from PG&E and NREL. Finally, our algorithm can perfectly reconstruct the topology in a short time, with and without the integration of DERs.

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